UDC 531.3

STABILITY OF NEUTRAL SYSTEMS WITH A DEVIATING ARGUMENT

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In furthering the results obtained earlier in [1, 2] general stability theorems using sign-constant Liapunov functions are established for neutral nonlinear systems with a deviating argument. Concepts connected with the stability of auxiliary difference inequalities play on essential role. The application of Liapunov's methods to the study of the stability of systems of neutral equations [5] has been examined in a number of papers [1 — 4]. A method for studying the stability of neutral systems by using only sign-constant but not sign-definite Liapunov functionals was proposed in [1]. This method was used in [2] to obtain stability conditions for first-order neutral nonlinear equations. An analogous method was used in [4] to investigate the stability of solutions of neutral equations with a linear neutral part.

1. Let R^n denote an n-dimensional Euclidean space with norm $|\cdot|$. By $C[a, b]$ we denote the space of functions $x(s)$ continuous on $[a, b]$, $x: R^1 \to R^n$, with the norm

$$\|x(s)\| = \max_{a \leqslant s \leqslant b} |x(s)|$$

Let $h > 0$ be some fixed number and $x(t) \in C[-h, T]$. By x_t we denote an element of space $C[-h, 0]$ of the form

$$x_t = x_t(\theta) = x(t + \theta), \quad -h \leqslant \theta \leqslant 0, \ t \in [0, T]$$

By S_H we denote a sphere in space $C[-h, 0]$

$$S_H = \{x(\theta) \in C[-h, 0], \|x(\theta)\| \leqslant H\}$$

Let

$$F(t, x_t), \quad F: [0, \infty) \times S_H \to R^n$$

$$G(t, x_t), \ G: [0, \infty) \times S_H \to R^n$$

be two prescribed continuous mappings, where for some $H > 0$

$$|F(t, x_t)| \leqslant M, \ t \in [0, \infty), \ x_t(\theta) \in S_H \tag{1.1}$$

We consider the following initial problem for the neutral functional-differential equation

$$d/dt\,[x(t) - G(t, x_t)] = F(t, x_t), \ x_0(\theta) = \varphi(\theta) \tag{1.2}$$

For any function $\varphi(\theta) \in C[-h, 0]$ we call $x(t) = x(t, \varphi)$ a solution of problem (1.2) on the interval $[0, \alpha]$, $\alpha > 0$, if $x(t) \in C[-h, \alpha]$, $x_0(\theta) = \varphi(\theta)$ and the function

$$Z(t, x_t) = x(t) - G(t, x_t)$$

has a continuous derivative satisfying Eq. (1.2) for each $t \in [0, \alpha]$. Everywhere below we assume that the solution of problem (1.2) exists and is unique. To ensure this we need to impose some further requirements on $G(t, x_t)$ and $F(t, x_t)$, in addition to the conditions already stated. For example, it is sufficient to assume that $G(t, \varphi)$ satisfies a local Lipschitz condition with a constant less than one for all functions $\varphi(\theta)$ and $\psi(\theta)$ such that $\varphi(s) \equiv \psi(s)$, $-h \leqslant s \leqslant -\varepsilon < 0$, for some $\varepsilon > 0$ and that $F(t, \varphi)$ satisfies a Lipschitz condition in sphere S_H. The exact statement of the existence an uniqueness theorem for problem (1.2) can be found in [6].

Let $G(t, 0) \equiv 0$ and $F(t, 0) \equiv 0$. Then Eq. (1.2) has a trivial solution corresponding to the initial function $\varphi(\theta) \equiv 0$.

D e f i n i t i o n 1. The trivial solution of problem (1.2) is:

a) stable if for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $|x(t, \varphi)| \leqslant \varepsilon$ when $t \geqslant 0$, if only $\|\varphi(\theta)\| \leqslant \delta(\varepsilon)$;

b) asymptotically stable if it is stable and, in addition,

$$\lim_{t\to\infty} x(t, \varphi) = 0$$

for all $\varphi(\theta) \in \Omega \subset C[-h, 0]$. The domain Ω is called the domain of attraction of the trivial solution.

In what follows a large role is played by concepts connected with the stability of difference inequalities. Consider the difference inequality

$$|Z(t, y_t)| = |y(t) - G(t, y_t)| \leqslant f(t), \quad y_0 = \varphi \tag{1.3}$$

Here $f(t)$ is a nonnegative continuous scalar function and $\varphi(\theta) \in C[-h, 0]$. By $y(t, \varphi)$ we denote the solution of difference inequality (1.3) with initial condition $y_0 = \varphi$. Remember that $G(t, 0) \equiv 0$.

D e f i n i t i o n 2. The trivial solution $y(t) \equiv 0$ of difference inequality (1.3) is:

a) f-stable if for any $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ such that $|y(t, \varphi)| \leqslant \varepsilon$ for all $t \geqslant 0$ under all initial conditions and right hand sides such that

$$\|\varphi(\theta)\| \leqslant \delta(\varepsilon), \quad \sup_{0 \leqslant t} f(t) \leqslant \delta(\varepsilon) \tag{1.4}$$

b) asymptotically f-stable if it is f-stable and, in addition,

$$\lim_{t\to\infty} y(t, \varphi) = 0 \tag{1.5}$$

for all $\varphi \in \Omega \subset C[-h, 0]$ and for every right hand side $f(t)$ such that $f(t) \to 0$ as $t \to \infty$;

c) f-bounded if a bounded solution $y(t, \varphi)$ corresponds to each bounded function $f(t)$.

Let $V(Z(t, x_t), x_t, t)$ be some functional, defined and continuous in all arguments for all $x_t \in S_H$ and $t \in [0, \infty)$, such that its derivative relative to Eq. (1.2) exists. We denote it

$$\frac{dV}{dt} = \frac{dV(Z(t, x_t), x_t, t)}{dt}$$

Since the derivative $dZ(t, x_t)/dt$ exists, while the derivative dx/dt may not exist, the requirement that the derivative dV/dt exist imposes definite constraints on the dependency of V on x_t. By $\omega_i(u)$, ω_i: $R^1 \to R^1$, we denote certain continuous nondecreasing functions such that

$$\omega_i (0) = 0; \ \omega_i (u) > 0, \quad u > 0 \tag{1.6}$$

T h e o r e m 1. Let a functional $V(Z(t, x_t), x_t, t)$ satisfying the requirements stated above exist and be such that

1) $\omega_1 (| Z(t, x_t) |) \leqslant V(Z(t, x_t), x_t, t) \leqslant \omega_2 (\| x_t (\theta) \|)$

2) $dV / dt \leqslant 0$

3) the trivial solution of difference inequality (1.3) is f-stable. Then the trivial solution of Eq. (1.2) is stable.

P r o o f. We take an arbitrary ε, $0 < \varepsilon < H$. By virtue of the f-stability of inequality (1.3), for this ε we can find $\delta_1 > 0$ such that every solution of the inequality

$$| Z(t, x_t) | = | x(t) - G(t, x_t) | \leqslant f(t), \quad x_0 = \varphi$$

will satisfy the relation

$$| x(t, \varphi) | \leqslant \varepsilon, \quad t \geqslant 0$$

under the conditions that $f(t) \leqslant \delta_1$, $t \geqslant 0$, $\| \varphi (\theta) \| \leqslant \delta_1$. We now take δ_2, $0 < \delta_2 \leqslant \delta_1$, such that $\omega_2 (\delta_2) = \omega_1 (\delta_1)$. Then when $\| \varphi (\theta) \| \leqslant \delta_2$, by virtue of conditions 1) and 2) we have

$$\omega_1 (| Z(t, x_t) |) \leqslant V(Z(t, x_t), x_t, t) \leqslant$$
$$V(Z(0, \varphi), \varphi, 0) \leqslant \omega_2 (\delta_2) = \omega_1 (\delta_1)$$

Hence, because of the monotonicity of function $\omega_1 (u)$ it follows that

$$| Z(t, x_t) | = f(t) \leqslant \delta_1$$

Allowing for the f-stability of the difference inequality, we get that $| x(t, \varphi) | \leqslant \varepsilon$, $t \geqslant 0$, for all $\varphi (\theta)$ such that $\| \varphi (\theta) \| \leqslant \delta_2 \leqslant \delta_1$. The theorem is proved.

T h e o r e m 2. Let a functional $V(Z(t, x_t), x_t, t)$ satisfying the hypotheses of Theorem 1 exist and be such that

$$dV(Z(t, x_t), x_t, t) / dt \leqslant -\omega_3 (| Z(t, x_t) |) \tag{1.7}$$

Let the trivial solution of difference inequality (1.3) be asymptotically f-stable. Then the trivial solution of Eq. (1.2) is asymptotically stable.

P r o o f. By virtue of Theorem 1 the trivial solution of Eq. (1.2) is stable. Hence we can find δ such that $x(t, \varphi) \in S_H$, $t \geqslant 0$, when $\varphi (\theta) \in S_\delta$. Let us show that then

$$| Z(t, x_t) | \to 0, \quad t \to \infty \tag{1.8}$$

Let this not be so. Then there would exist a number γ, $0 < \gamma < H$, and a sequence of points $t_i \to \infty$ such that

$$| Z(t_i, x_{t_i}) | \geqslant \gamma$$

by Eq. (1.2) and condition (1.1) we have

$$| dZ(t, x_t) / dt | = | F(t, x_t) | \leqslant M$$

when $\varphi \in S_\delta$. Hence,

$$| Z(\tau, x_\tau) | \geqslant \gamma / 2$$

for $\tau \in [t_i - \gamma (2M)^{-1}, t_i + \gamma (2M)^{-1}]$. We denote the number of points $t_i \in [0, t]$ by $n(t)$. Then because of (1.7) we have

$$V(Z(t, x_t), x_t, t) - V(Z(0, \varphi), \varphi, 0) = \int_0^t \frac{dV}{ds} ds \leqslant$$
$$- \int_0^t \omega_3(|Z(s, x_s)|) ds \leqslant - \frac{\gamma}{M} \omega_3\left(\frac{\gamma}{2}\right) n(t) \to -\infty, \quad t \to \infty$$

This, however, contradicts the fact that the difference $V(Z(t, x_t), x_t, t) - V(Z(0, \varphi), \varphi, 0)$ is bounded by virtue of condition 1). Hence, we have proved that relation (1.8) is valid when $\varphi(\theta) \in S_\delta$. From this, by virtue of the asymptotic f-stability of inequality (1.3), follows the asymptotic stability of the trivial solution of Eq. (1.2). Theorem 2 is proved.

As a corollary of Theorem 2 we establish the following statement. Consider the ordinary differential equation

$$x^{\cdot}(t) = F(t, x(t)), \quad x(t_0) = x_0, \quad t_0 \geqslant 0 \tag{1.9}$$

Here F: $[0, \infty) \times R^n \to R^n$ is a functions continuous in all its arguments, satisfying a local Lipschitz condition in the second argument; $F(t, 0) \equiv 0$. Together with Eq. (1.9) consider the neutral functional-differential equation

$$dZ(t, x_t) / dt = F(t, Z(t, x_t)), \quad x_0 = \varphi \tag{1.10}$$
$$Z(t, x_t) = x(t) - G(t, x_t)$$

where $G(t, x_t)$ satisfies the conditions stated above.

T h e o r e m 3. Let the trivial solution of Eq. (1.9) be uniformly asymptotically stable with respect to the initial instant t_0 and the initial coordinate x_0. Further, let the trivial solution of difference inequality (1.3) be asymptotically f-stable. Then the trivial solution of Eq. (1.10) is asymptotically stable.

P r o o f. By virtue of the inversion theorem [7], for Eq. (1.9) there exists a continuously differentiable Liapunov function $W(t, x)$ such that

$$\omega_1(|x|) \leqslant W(t, x) \leqslant \omega_2(|x|)$$
$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \sum_{i=1}^n \frac{\partial W}{\partial x_i} \frac{dx_i}{dt} \leqslant - \omega_3(|x(t)|)$$

We now consider the functional $W(t, Z(t, x_t))$. It is easy to see that

$$\omega_1(|Z(t, x_t)|) \leqslant W(t, Z(t, x_t)) \leqslant \omega_2(|Z(t, x_t)|) \leqslant$$
$$\omega_4(\|x_t\|)$$
$$\frac{dW}{dt} = \frac{\partial W}{\partial t} + \sum_{i=1}^n \frac{\partial W}{\partial Z_i} \frac{dZ_i}{dt} \leqslant - \omega_3(|Z(t, x_t)|)$$

when $x_t \in S_H$. Therefore, functional $W(t, Z(t, x_t))$ satisfies all the hypotheses of Theorem 2. Allowing for the assumed asymptotic f-stability of the trivial solution of difference inequality (1.3), we get that the trivial solution of the neutral Eq. (1.10) is asymptotically stable. Theorem 3 is proved.

T h e o r e m 4. Let the mapping $G(t, x_t)$ satisfy the Lipschitz condition

$$|G(t, x_t) - G(t, y_t)| \leqslant \alpha \|x_t - y_t\|, \quad 0 < \alpha < 1 \tag{1.11}$$

$$\forall x_t,\ y_t \in S_H,\quad 0 \leqslant t < \infty$$

Let a function $V\ (Z\ (t,\ x_t),\ x_t,\ t)$ exist, satisfying condition 1) of Theorem 1 and such that

$$dV\ /\ dt \leqslant -\ \omega_5\ (|\ x\ (t)\ |\ ) \tag{1.12}$$

Then the trivial solution of Eq. (1.2) is asymptotically stable.

P r o of. By virtue of Theorem 1 and Lemma 1, proved below, the trivial solution of Eq. (1.2) is stable. Therefore, $x\ (t,\ \varphi) \in S_H$ for $\varphi\ (\theta) \in S_\delta$. Let us show tnat $x\ (t) \to 0$ as $t \to \infty$ and thus prove the asymptotic stability. Assume the contrary. Then a number $\gamma > 0$ and a sequence of points t_i exist such that $|\ x\ (t_i)\ | > \gamma$. From Eq. (1.2) follows

$$x(t) = G\,(t,\ x_t) + \int_0^t F\,(s,\ x_s)\, ds$$

Here we assume that $x\ (s)\ =\ \varphi\ (s)$ when $s \leqslant 0$. Then, allowing for (1.11), for $\Delta > 0$ we have

$$|\, x\,(t+\Delta) - x\,(t)\,| = |\, G\,(t+\Delta,\ x_{t+\Delta}) - G\,(t,\ x_t) + \int_t^{t+\Delta} F\,(s,\ x_s)\, dt\,| \leqslant \alpha\,\|\, x_{t+\Delta} - x_t\,\| + M\Delta \tag{1.13}$$

We denote

$$\rho\ (\eta)\ =\ \sup_{0\leqslant t}\ \sup_{\Delta\leqslant\eta}\ |\ x\ (t\ +\ \Delta)\ -\ x\ (t)\ |$$

From (1.13) follows

$$\rho\ (\eta) \leqslant \alpha\,\rho\ \ (\eta) + \alpha \max_{\eta\geqslant\Delta}\ \max_{-h\leqslant\theta\leqslant 0}\ |\ x\ (\theta\ +\ \Delta)\ -\ \varphi\ (\theta)\ |\ +\ M\eta$$

Hence

$$\rho\,(\eta) \leqslant \frac{\alpha}{1-\alpha} \max_{\eta\geqslant\Delta}\ \max_{-h\leqslant\theta\leqslant 0} |\,x\,(\theta+\Delta) - \varphi\,(\theta)\,| + \frac{M}{1-\alpha}\,\eta$$

The right-hand side of this inequality tends to zero as $\eta \to 0$ because the function $x\ (\theta)$ is uniformly continuous on the closed interval $[-h,\ \eta]$. Hence we can find $\bar\eta > 0$ such that $\rho\ (\eta) \leqslant \gamma/2$. In this connection, $|\ x\ (\tau)\ | \geqslant \gamma\ /\ 2$ for $\tau \in [t_i - \bar\eta,\ t_i + \bar\eta]$ uniformly over all i . Just as in Theorem 2 a contradiction follows at once from this. Theorem 4 is proved.

Concrete stability conditions in terms of the coefficients of first-order scalar equations with distributed and unbounded deviating arguments, which may be looked upon as an immediate corollary of Theorem 3, were obtained in [3].

2. Let us now consider the question of stability in-the-large of the trivial solution of Eq. (1.2). We shall take it that the mappings $F\ (t,\ x_t)$ and $G\ (t,\ x_t)$ are defined and continuous on the whole space $[0,\ \infty) \times C\ [-\ h,\ 0]$. In addition, let $|\ F\ (t,\ x_t)\ | \leqslant M_H$ for all $H > 0,\ x_t \in S_H$ and $t \geqslant 0$. We assume as well that the hypotheses of the existence and uniqueness theorem are fulfilled for an arbitrary sphere S_H.

D e f i n i t i o n 3. The trivial solution of Eq. (1.2) is asymptotically stable in-the-large if it is stable and the equality

$$\lim_{t\to\infty}\ |\ x\ (t,\ \varphi)\ |\ =\ 0 \tag{2.1}$$

is valid for any initial function $\varphi(\theta)$.

Theorem 5. Let a functional $V(Z(t, x_t), x_t, t)$ exist, defined and continuous on the whole space $[0, \infty) \times C[-h, 0]$ and satisfying all the hypotheses of Theorem 2 or 4 on the whole space. In addition, let

$$\omega_1(u) \to \infty, \quad u \to \infty \tag{2.2}$$

and let the trivial solution of difference inequality (1.3) be f-bounded. Then the trivial solution of Eq. (1.2) is asymptotically stable in-the-large.

Proof. The trivial solution of Eq. (1.2) is obviously stable. Let us prove that the second condition in Definition 3 is fulfilled. To do this we show first of all that any solution $x(t, \varphi)$ is bounded. Let $\|\varphi(\theta)\| = K$. By virtue of (2.2) we can find $R \geqslant K > 0$ such that

$$\omega_1(R) = \omega_2(K), \quad \omega_1(u) > \omega_2(K) \ \text{ for } \ u > R.$$

In this connection,

$$|Z(t, x_t(t+\theta, \varphi)| \leqslant R, \quad t \geqslant 0 \tag{2.3}$$

In the contrary case, i.e., when $s \geqslant 0$ exists such that $|Z(s, x_s)| > R$, there would hold the opposite relation

$$\omega_2(K) = \omega_1(R) < \omega_1(|Z(s, x_s)|) \leqslant V(Z(s, x_s), x_s, s) \leqslant V(Z(0, \varphi), \varphi, 0) \leqslant \omega_2(K)$$

Hence, inequality (2.3) has been established. By the assumption of the f-boundedness of the solutions of inequality (2.3) we get that we can find $q > 0$ such that

$$|x(t, \varphi)| \leqslant q, \quad t \geqslant 0$$

All the hypotheses of Theorem 2 or 4 are fulfilled in sphere S_q; therefore, relation (2.1) is valid. Theorem 5 is established.

3. Let us set up certain sufficient tests for the f-stability of the trivial solution of difference inequality (1.3).

Lemma 1. Let mapping $G(t, y_t)$ satisfy Lipschitz condition (1.11). Then the trivial solution of inequality (1.3) is f-stable and f-bounded.

Proof. From inequality (1.3) follows

$$|y(t)| \leqslant |G(t, y_t)| + f(t) \leqslant \alpha \|y_t(\theta)\| + f(t)$$

We denote

$$m(t) = |\sup_{0 \leqslant s \leqslant t} \|y(s)|)$$

Then we obtain

$$m(t) \leqslant \alpha\, m(t) + \alpha \|\varphi(\theta)\| + \sup_{0 \leqslant s \leqslant t} f(s)$$

Hence it follows that

$$m(t) \leqslant \frac{\alpha}{1-\alpha} \|\varphi(\theta)\| + \frac{1}{1-\alpha} \sup_{0 \leqslant s \leqslant t} f(s) \tag{3.1}$$

Inequality (3.1) signifies that the trivial solution of (1.3) is f-stable and f-bounded.

Lemma 2. Let

$$G(t, y_t) \equiv g(t, y(t-h))$$

where g is a function from $R^1 \times R^n$ into R^n and $h > 0$ is some number. Then, if

$$| g(t, y(t-h)) | \leqslant \gamma \, | y(t-h) |, \quad 0 < \gamma < 1 \tag{3.2}$$

then the trivial solution of inequality

$$| y(t) - g(t, y(t-h)) | \leqslant f(t) \tag{3.3}$$

is asymptotically f-stable.

Proof. From (3.2) and (3.3) follows

$$| y(s+h) | \leqslant | g(s, y(s)) | + f(s+h) \leqslant \gamma \, | y(s) | + f(s+h) \tag{3.4}$$

for $-h \leqslant s \leqslant 0$. Analogously, we obtain

$$| y(s+nh) | \leqslant \gamma \, | y(s+(n-1)h) | + f(s+(n-1)h) \leqslant \gamma^n | y(s) | + f(s+(n-1)h) + \gamma f(s+(n-2)h) + \ldots + \gamma^{n-1} f(s+h)$$

By virtue of the f-stability of the trivial solution of (3.3) we can obviously find an N such that

$$\gamma^n | y(s) | \leqslant \varepsilon/3 \quad \text{for } n \geqslant N \tag{3.5}$$

Further, since $f(t)$ is a bounded function tending to zero, a number m exists such that

$$f(s+nh) \leqslant \frac{\varepsilon}{3}(1-\gamma), \quad \frac{\gamma^m}{1-\gamma} \max_{t \geqslant 0} f(t) \leqslant \frac{\varepsilon}{3}$$

when $n \geqslant m$. But then for $n \geqslant 2m$ we have

$$\begin{aligned} & f(s+nh) + \ldots + \gamma^{n-1} f(s+h) = f(s+nh) + \gamma f(s+(n-1)h) + \\ & \ldots + \gamma^{n-m+1} f(s+mh) + \gamma^{n-m} [f(s+(m-1)h) + \ldots \\ & \quad + \gamma^{m-1} f(s+h)] \max_{t \geqslant 0} f(t) \leqslant \\ & \frac{\varepsilon}{(1-\gamma)3}(1-\gamma) + \frac{\gamma^m}{1-\gamma} \max_{t \geqslant 0} f(t) \leqslant \frac{2}{3}\varepsilon \end{aligned} \tag{3.6}$$

Inequalities (3.5) and (3.6) are simultaneously fulfilled when $n \geqslant \max\{N, 2m\}$, i.e.,

$$| y(s+nh) | \leqslant \varepsilon, \quad -h \leqslant s \leqslant 0$$

Hence, $\| y_{nh}(\theta) \| \leqslant \varepsilon$.

In order to state Lemma 3 we present certain concepts from the theory of almost-periodic functions [8]. The spectrum of an almost-periodic function $\varphi(t)$ is the set

$$\Lambda(\varphi) = \{\lambda \in R^1 : M[e^{i\lambda t} \varphi(t)] \neq 0\}$$

$$M[\psi] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \psi(t)\, dt$$

The set

$$\mathrm{Mod}(\varphi) = \left\{ \sum_{i=1}^{K} m_i \lambda_i, \; \lambda_i \in \Lambda(\varphi), \; m_i \text{ are integers}, \; K \text{ is a positive integer} \right\}$$

is called the modulus of the almost-periodic function.

Lemma 3. Let

$$\dot{G}(t, y_t) \equiv g(t, y(t-h)), \quad g: R^1 \times R^n \to R^n, \; h > 0$$

$$| g(t, y(t-h)) | \leqslant \gamma(t) \, | y(t-h) |, \quad \gamma(t) > 0.$$

Let the function $\gamma(t)$ either be almost-periodic and satisfy the conditions

$$\forall(\lambda:\lambda\in \mathrm{Mod}(\gamma),\ \lambda\neq 0)\ [h\lambda\neq 0 \pmod{2\pi}]$$

$$M(\ln\gamma(t))<0$$

or be ω-periodic, h be incommensurable with ω and the inequality

$$\int_0^\omega \ln\gamma(t)\,dt<0$$

be valid. Then the trivial solution of inequality (3.3) is asymptotically f-stable and f-bounded.

P r o o f. Consider the operator

$$(Ay)(t)=\gamma(t)\,y(t-h)$$

As was shown in [9], when the hypotheses of Lemma 3 are fulfilled the spectra radius $r(A)$ of operator A is less than unity

$$r(A)=\lim_{n\to\infty}(\|A^n\|)^{1/n}<1$$

Hence, for a sufficiently large l

$$\gamma_1=\|A^l\|<1$$

But then

$$\begin{aligned}|y(s+lh)|&\leqslant\gamma(s+lh)\,|y(s+(l-1)h)|+f(s+lh)=\\ &(Ay)\ (s+lh)+f(s+lh)\leqslant\ldots\leqslant(A^ly)(s+lh)+\ f(s+lh)+\\ &(Af)(s+lh)\ +(A^2f)(s+lh)+\ldots+(A^{l-1}y)(s+lh)=\\ &(A^ly)(s+lh)+F_1(s+lh)\leqslant\gamma_1|y(s)|+F_1(s+lh)\end{aligned}$$

Here we have set

$$F_1(s+lh)=f(s+lh)+(Af)(s+lh)+\ldots+(A^{l-1}f)\ (s+lh)$$

where it is evident that $F_1(t)$, just as $f(t)$, is a bounded function tending to zero as $t\to\infty$. Denoting $lh=h_1$, we obtain the inequality

$$|y(s+h_1)|\leqslant\gamma_1|y(s)|+F_1(s+h_1),\quad 0<\gamma_1<1$$

completely analogous to inequality (3.4). If now we repeat the end part of the proof of Lemmas 1 and 2, we obtain the assertion of Lemma 3.

Suppose now that mapping $G(t, y_t)$ does not depend upon the values of function $y_t(\theta)$ when $\theta\in[t-\Delta, t]$. Here Δ, $0<\Delta<h$ is a prescribed number. We note that in this case we apply the step method for solving difference inequality (1.3).

L e m m a 4. Let $G(t, y_t)$ be independent of the values of $y_t(\theta)$, $\theta\in[t-\Delta, t]$, and satisfy Lipschitz condition (1.11). Further, let

$$\alpha+\alpha^2+\ldots+\alpha^N=\gamma_3<1 \tag{3.7}$$

Here N is the smallest positive integer such that $N\Delta\geqslant h$. Then the trivial solution of inequality (1.3) is asymptotically f-stable.

P r o o f. From relations (1.3) and (1.11) follows the bound

$$\max_{0\leqslant t\leqslant\Delta}|y(t)|\leqslant\max_{0\leqslant t\leqslant\Delta}\{|G(t, y_t)|+f(t)\}\leqslant\alpha\,\|\varphi(\theta)\|+\max_{0\leqslant t\leqslant\Delta}\ f(t)$$

Further,

$$\max_{0\leqslant t\leqslant 2\Delta}|y(t)|\leqslant\alpha\ \{\max_{0\leqslant t\leqslant\Delta}|y(t)|+\|\varphi(\theta)\|\}+\ \max_{0\leqslant t\leqslant 2\Delta}f(t)\leqslant(\alpha+$$

$$\alpha^2) \| \varphi(\theta) \| + \alpha \max_{0 \leqslant t \leqslant \Delta} f(t) + \max_{0 \leqslant t \leqslant 2\Delta} f(t)$$

Analogously we have

$$\max_{0 \leqslant t \leqslant h} | y(t) | = \| y_h(\theta) \| \leqslant (\alpha + \alpha^2 + \ldots + \alpha^N) \| \varphi(\theta) \| + F_2(h) \qquad (3.8)$$

$$F_2(t) = \max_{t-h \leqslant s \leqslant t} f(s) + \alpha \max_{t-h+\Delta \leqslant s \leqslant t} f(s) + \ldots +$$

$$\alpha^N \max_{t-h+(N-1)\Delta \leqslant s \leqslant t} f(s)$$

The function $F_2(t)$, just as $f(t)$, is bounded and tends to zero as $t \to \infty$. We note that inequality (3.8) is similar to inequality (3.4). We can now convince ourselves of the validity of Lemma 4's assertion by repeating the end part of the proof of Lemma 2.

Note. Condition (3.7), occurring in the statement of Lemma 4, is fulfilled, for example, if $\alpha < 1/2$.

4. Examples.

1°. Consider the equation

$$x^{\cdot\cdot}(t) + cx^{\cdot\cdot}(t-h) + g(x(t) + cx(t-h)) = 0, \ h > 0 \qquad (4.1)$$

Under the conditions that

$$ug(u) > 0, \ u \neq 0, \ |c| < 1$$

the trivial solution of (4.1) will be stable. We write (4.1) as

$$Z(t, x_t) = x(t) + cx(t-h)$$

$$Z^{\cdot} = \omega, \ \dot{\omega} = -g(Z)$$

and we consider the functional

$$V(Z(t, x_t), \omega) = \frac{\omega^2}{2} + \int_0^Z g(u)\, du \qquad (4.2)$$

The derivative of functional (4.2) on the solutions of Eq. (4.1) equals $dV/dt = \omega\dot{\omega} + Zg^{\cdot}(Z) = 0$. By virtue of Theorem 1 and Lemma 1, all of whose hypotheses are fulfilled, the trivial solution of Eq. (4.1) is stable

2°. Consider the equations

$$x^{\cdot\cdot}(t) + \varphi(x(t))\, x^{\cdot}(t) + f(x) = 0 \qquad (4.3)$$

$$Z^{\cdot\cdot}(t, x_t) + \varphi(Z(t, x_t))\, Z^{\cdot}(t, x_t) + f(Z(t, x_t)) \qquad (4.4)$$

$$Z(t, x_t) = x(t) + {}^1/_2 \exp(\sin t)\, x(t-1)$$

Under the conditions that

$$x f(x) > 0, \quad x \neq 0, \quad \varphi(x) > 0$$

the trivial solution of Eq. (4.3) is uniformly asymptotically stable [10]. By virtue of Theorem 3 and Lemma 3 the trivial solution of Eq. (4.4) too is asymptotically stable. Moreover if

$$\int_0^x f(s)\, ds \to \infty, \quad |x| \to \infty$$

then the trivial solution of (4.4) is asymptotically stable in-the-large.

REFERENCES

1. Kolmanovskii, V. B. and Nosov, V. R., On the stability of nonlinear oscillating systems described by neutral equations. Proc. Fifth Internat. Conf. Nonlinear Oscillations, Kiev, 1969, Vol. 2. Kiev, Izd. Inst. Mat. Akad. Nauk Ukr. SSR, 1970.

2. Kolmanovskii, V. B. and Nosov, V. R., On the stability of first-order nonlinear equations of neutral type. PMM Vol. 34, No. 4, 1970.

3. Misnik, A. F. and Nosov, V. R., On the stability of neutral differential-difference equations. In: Collection of Scientific Papers of Graduate Students at the Lumumba University of the Friendship of Peoples, No. 1. Moscow, 1968.

4. Cruz, M. A., and Hale, J. K. Stability of functional differential equations of neutral type. J. Different. Equat., Vol. 7, No. 2, 1970.

5. El'sgol'ts, L. E., Introduction to the Theory of Differential Equations with a Deviating Argument. Moscow, "Nauka", 1964. (see also English translation, Pergamon Press, Book No. 09554, 1962, Distributed in the U. S. A. by Addison-Wesley Publ. Co.).

6. Hale, J. K., and Cruz, M. A. Existence, uniqueness and continuous dependence for hereditary systems. Ann. matem. pura ed appl. No. 85, 1970.

7. Krasovskii, N. N., Certain Problems in the Theory of Stability of Motion. Moscow, Fizmatgiz, 1959. (see also English translation, Stanford University Press, Stanford, California, 1963).

8. Levitan, B. M., Almost-Periodic Functions. Moscow, Gostekhizdat, 1953.

9. Mukhamadiev, E. and Sadovskii, B. N., On an estimate of the spectral radius of an operator connected with neutral equations. Mat. Zametki, Vol. 13, No. 1, 1973.

10. Barbashin, E. A., Liapunov Functions, Moscow, "Nauka", 1970.

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